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SOME NEW MATHEMATICAL METHODS ON QUANTUM THEORY

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Abstract

First, iso-mathematics and its wide applications constitute iso-science, which includes physics, quantum theory, chemistry, energy and cosmology, etc. Second, high dimensions of mathematics and physics, and basic equation of energy-momentum representation are researched. Third, the nonlinear quantum theory is discussed. Fourth, possible unified equation and theory of fermions and bosons are proposed. Fifth, mathematical base of the extensive quantum theories is studied. Finally, some possible developments of quantum equations are searched.

Key words: quantum theory, mathematics, iso-science, equation, unification.

I. Iso-Mathematics and Its Applications

Physics and mathematics are always closely related and promote each other. From 1978 Santilli discovered the new Iso-Mathematics, which includes isonumbers, isorepresentation [1-3], etc. In Iso-Mathematics, isonumbers and genonumbers of dimensions 1, 2, 4, 8, and their isoduals and pseudoduals may extend to "hidden numbers" of dimension 3, 5, 6, 7 [2]. They are applied for many regions [3,4].

In 1998 Santilli shown that the objections against the Einstein-Podolsky-Rosen (EPR) argument are valid for point-like particles in vacuum (*exterior dynamical systems*), but the same objections are inapplicable (rather than being violated) for extended particles within hyperdense physical media (*interior dynamical systems*) because the latter systems appear to admit an identical classical counterpart when treated with the isotopic branch of hadronic mathematics and mechanics. Now Santilli reviewed, upgraded and specialized the basic mathematical, physical and chemical methods for the study of the EPR prediction that quantum mechanics is not a complete theory. This includes basic methods [5], apparent proof of the EPR argument [6], and examples and applications, in which the validity of the EPR final statement is the effect that the wave function of quantum mechanics does not provide a complete description of the physical reality. The axiom-preserving "completion" of the quantum mechanical wave function due to deep wave-overlapping when represented via isomathematics, and shown that it permits an otherwise impossible representation of the attractive force between identical electrons pairs in valence coupling, as well as the representation of *all* characteristics of various physical and chemical systems existing in nature [7]. Moreover, Santilli studied the classical determinism of EPR prediction by isomathematics [8].

Santilli studied generalization of Heisenberg's uncertainty principle for strong interactions [9]. Santilli searched isorepresentation of the Lie-isotopic SU(2) algebra with application to nuclear physics and local realism [10], and studied the classical determinism predicted by A. Einstein, B. Podolsky and N. Rosen [8]. Santilli discussed foundation of theoretical mechanics [11,12], because of contact, thus continuous and instantaneous character, by therefore voiding the need for superluminal communications.

The non-linear, non-local and non-potential character of the assumed interactions render them ideally suited for their representation via the isotopic (i.e. axiom-preserving branch of hadronic mechanics [13-15]), comprising iso-mathematics and iso-mechanics, which are characterized by the isotopy of the

universal enveloping associative algebra ξ of quantum mechanical Hermitian operators A, B, ... with isoproduct, and studied Einstein-Podolsky-Rosen prediction [5-7].

Santilli first recalled the 1935 historical view by A. Einstein, B. Podolsky and N. Rosen according to which "Quantum mechanics is not a complete theory" (EPR argument), because of the inability by quantum mechanics to provide a quantitative representation of the interactions occurring in particle entanglements. They then shown, apparently for the first time, that the completion of quantum entanglements into the covering EPR entanglements formulated according to hadronic mechanics provides a quantitative representation of the interactions occurring in particle entanglements by assuming that their continuous and instantaneous communications at a distance are due to the overlapping of the wave packets of particles, and therefore avoiding superluminal communications. According to this view, entanglement interactions result to be nonlinear, non-local and not derivable from a potential, and are represented via Bohm's variable λ hidden in the quantum mechanical associative product of Hermitean operators AB=A×B via explicit and concrete, axiom-preserving realizations $A\hat{\times}B = A\lambda B$, $I\hat{\times}A = A\hat{\times}I = A$ with ensuing non-unitary structure, multiplicative unit $U/U^{+} = \hat{I} = 1/\lambda$, inapplicability' of Bell's inequalities and consequential validity of Bohm's hidden variables. We finally introduce, also apparently for the first time, the completion of quantum computers into the broader EPR computers characterizing a collection of extended electronic components under continuous entanglements, and show their apparent faster computation, better cybersecurity and improved energy efficiency [16].

More explicitly, the quantum mechanical equation for two interacting particles with coordinates r_k (k=1,2) on a Hilbert space H over the field C of complex numbers is given by the familiar Schrodinger equation [16]:

$$[\sum_{k=1,2}\frac{1}{2m_k}(p_k p_k + V(r))]\psi(r) = E\psi(r). \qquad (1)$$

By recalling the basic expression of the isolinear iso-momentum characterized by the completion of the local Newton-Leibnitz differential calculus into the non-local iso-differential calculus [20]:

$$i\hat{I}\frac{\partial}{\partial t}\psi(r) = i\frac{1}{\lambda}\frac{\partial}{\partial(r/\lambda)}\psi(r/\lambda). \qquad (2)$$

The non-relativistic version of the EPR entanglement is characterized by the iso-Schrodinger equation.

The new entanglement interaction verifies, by conception and construction, the abstract axioms of relativistic quantum mechanics although realized via the indicated universal iso-associative envelope [17,18].

Santilli researched invariant Lie-isotopic and Lie-admissible formulation of quantum deformations [19], and new conception of living organisms and its representation via Lie-admissible Hv-hyperstructures [20]. Santilli proposed foundations of hadronic chemistry and applied to new clean energies and fuels [21]. Santilli and Shillady discussed a new isochemical model of the hydrogen molecule [22]. Faster computations, since all values of Bohm hidden variable λ are very small according to all available fits of experimental data, with ensuing rapid convergence of iso-perturbative series. As a confirmation of this expectation, Santilli recalled the achievement via iso-mathematics and iso-chemistry of the first known attractive force between the identical electrons of valence coupling (see Chapter 4 of [21]), resulting in a strong valence bond that allowed the first known numerically exact representation of the experimental data for the hydrogen [22] and water [23] molecules with iso-perturbative calculations at least one thousand times faster than their conventional chemical counterparts.

Clear experimental evidence in various fields of deviations of physical reality from quantum predictions in favor of exact representations via hadronic mechanics, including deviations in nuclear physics [24]; electrodynamics [25,26]; nuclear physics [24]; condensed matter physics; heavy ion physics; time dilation for composite particles [27]; Bose-Einstein correlation [28]; cosmology [29]; and other fields. The violation of causality may explain the lack of achievement to date of controlled nuclear fusion.

Santilli proposed a unified form of possible generalizations of Heisenberg uncertainty principle for strong interactions [24]. Hadronic mechanics (HM) [25,26] has generalized the uncertainty relations in the only non-trivial way.

In 2023 Santilli reviewed and updated that the insufficiencies of quantum mechanics in nuclear physics; the completion of quantum mechanics into the axiom-preserving, Lie-isotopic branch of hadronic mechanics for the invariant representation of extended protons and neutrons under potential and contact/non-potential interactions; the exact hadronic representation of all characteristics of the neutron in its synthesis from the proton and the electron at the non-relativistic and relativistic levels; the completions of Bell's inequalities with ensuing iso-deterministic principle for strong interactions. Santilli then presented the apparent resolution of the historical objections against the reduction of all stable matter in the universe to protons and electrons and point

out a number of open problems whose treatment is beyond the capabilities of quantum mechanics, such as: the cosmological implications of the missing energy in the neutron synthesis, the prediction of negatively charged pseudo-protons, and the possible recycling of radioactive nuclear waste by nuclear power plants via their stimulated decay [30].

In a word, iso-mathematics and its wide applications constitute a new iso-science, which includes physics, quantum theory, chemistry, energy, cosmology and so on [1-30]. Further, in this paper we research some new developments of quantum theory from high dimensions of mathematics and physics, and the nonlinear quantum theory to possible unified equation and theory of fermions and bosons, etc.

2. Some Known Mathematical Methods in Quantum

In mathematical method in quantum physics, Hamiltonians may apply to classical mechanics and statistics, quantum mechanics and so on. In mathematics Hamiltonian is an operator in fact, and its extension (moment map, etc.) may apply to dynamical system, differential equation, Lie group, symplectic geometry, etc.

Some mathematical methods on quantum theory are always the important concern for physicists. Recently, Nakahira focused on identifying sudden changes in weak signals transmitted by electromagnetic or gravitational waves, and demonstrated quantum change points for Hamiltonians [31]. Carollo discussed non-Gaussian dynamics of quantum fluctuations and mean-field limit in open quantum central spin systems [32]. Jalali-mola, et al., searched topological Bogoliubov quasiparticles from Bose-Einstein Condensate in a flat band system [33].

In the standard quantum theory, the causal order of occurrence between events is prescribed, and must be definite. This has been maintained in all conventional scenarios of operation for quantum batteries. Zhu, et al., studied to allow the charging of quantum batteries in an indefinite causal order (ICO), and proposed a nonunitary dynamics-based charging protocol and experimentally investigate this using a photonic quantum switch. Their results demonstrate that both the amount of energy charged and the thermal efficiency can be boosted simultaneously. They found that ICO protocol can outperform the conventional protocols and gives rise to the anomalous inverse interaction effect, and provide new insights into ICO and its potential applications [34]. The quantum geometry has significant consequences in determining transport and optical properties in quantum materials. Kaplan, et al., used a semiclassical

formalism coupled with perturbative corrections unifying the nonlinear anomalous Hall effect and nonreciprocal magnetoresistance (longitudinal resistance) from the quantum geometry. They demonstrated the coexistence of a nonlinear anomalous Hall effect and nonreciprocal magnetoresistance in films of the doped antiferromagnetic topological insulator MnBi2Te4, and indicated that both longitudinal and transverse nonlinear transport provide a sensitive probe of the quantum geometry in solids [35]. Faugno, et al., investigated a discrete nonlinear Schrödinger equation with dynamical, density-difference-dependent gauge fields. They found a ground-state transition from a plane wave condensate to a localized soliton state as the gauge coupling is varied, and found a regime in which the condensate and soliton are both stable, and identified an emergent chiral symmetry, which leads to the existence of a symmetry-protected zero-energy edge mode. The emergent chiral symmetry relates low and high energy solitons. These states indicate that the interaction acts both repulsively and attractively [36].

3. High Dimensions and Basic Equation of Energy-Momentum Representation

The space-dimension of mathematics and physics can be extended to higher n-dimensions geometry [37]. It may be Hilbert space and corresponding quantum mechanics, and superstring. In mathematics, physics and many regions the field theory is all a very important problem. When the space-dimension is extended to n, fractal and complex, and various number-systems, the field theory and its formulas may be correspondingly extended. In these cases, Gauss's theorem and Stokes's theorem, and corresponding extensions on gradient, divergence and curl are searched [38]. Further, they are combined each other, and form multiple combinations, such as the scalar-tensor fields, the scalar-spinor fields, the vector-spinor fields, etc. These fields can be applied to physics, biology, earthquakes and social science, etc. Field theory has been widely applied in many regions of natural and social sciences, and its any development will necessarily inspire and apply to more aspects [39].

Based on the non-commutation

$$[A,B]=AB-BA=\eta\neq 0, \qquad (3)$$

of matrix, group, tensor and so on, we proposed the mathematical quantum theory. General matrix and quaternion are non-commutation, but special matrix may be commutation. If η is imaginary number, it will correspond to the extensive quantum theory. If η is real number, it will be development of quantum theory. Moreover, η may be complex number, etc. We introduced a

similar wave function and corresponding operators, various similar quantum results are derived. Further, we discussed its physical meaning and various applications. Based the general matrix we researched mathematics of unified gravitational and electromagnetic fields, and discussed the space-time equations and the simplest unifying quantum theory and general relativity. This can combine general discrete mathematics [40].

It is known that momentum-energy operators in quantum mechanics are:

$$p_\alpha = -i\hbar\frac{\partial}{\partial x_\alpha}, E = i\hbar\frac{\partial}{\partial t}. \qquad (4)$$

The four-dimensional equations are:

$$p_\mu \psi = -i\hbar\frac{\partial}{\partial x_\mu}\psi. \qquad (5)$$

In quantum mechanics the time-space operators of energy-momentum representation in quantum mechanics are [41,426]:

$$x_\mu \psi = i\hbar\frac{\partial}{\partial p_\mu}\psi. \qquad (6)$$

This includes the quantum equation in energy representation:

$$T\psi = -i\hbar\frac{\partial}{\partial E}\psi. \qquad (7)$$

And the space operator equation [43]:

$$r\psi = i\hbar\frac{\partial}{\partial p}\psi. \qquad (8)$$

Based on Eq.(6) we discussed some applications, in particular, the lifetime formulas [43].

Eq.(7) may extensive to:

$$i\hbar\frac{\partial}{\partial E}\psi = -(T+V)\psi. \qquad (9)$$

This may obtain the lifetime formulas. When potential is rotation or at spherical coordinates, we may obtain $\frac{l(l+1)}{r^2}$. Such the equation with the spherical potential in bound states is:

$$\frac{d^2}{dr^2}\psi - \frac{l(l+1)}{r^2}\psi + \frac{2m}{\hbar^2}[E - V(r)]\psi = 0. \qquad (10)$$

The total energy is:

$$E = E_0 + An + Bl(l+1). \qquad (11)$$

This is similar to diatomic molecules.

Eq.(10) corresponds to equation in the spherical potential:

$$[\frac{d^2}{dp^2}-\frac{l(l+1)}{p^2}+\frac{E-V}{\hbar^2}]\psi=0. \qquad (12)$$

$$\frac{d^2}{dE^2}\psi-\frac{l(l+1)}{E^2}\psi+[T(E)-V]\psi=0. \qquad (13)$$

This adds the harmonic oscillator and the anharmonic oscillator $V=aE^2+bE^3$, and $\tau=h/E$. In momentum representation, Eq.(15) becomes to:

$$[\frac{d^2}{dp^2}-\frac{l(l+1)}{p^2}+\frac{1}{\hbar^2}(E-ap^2-bp^3)]\psi=0.(14)$$

Planck formula was originally based on a series of harmonic oscillators. This should obtain different quantum lifetime. From the time equation (7)(9), we obtained some lifetime formulas [44-46], and they agree better with the experimental data [47].

Further, we proposed some operator equations of general relativity and special relativity, for example:

$$g_{\mu\nu}\frac{\partial^2}{\partial p_\mu \partial p_\nu}\psi+\frac{s^2}{\hbar^2}\psi=0. \qquad (15)$$

This is the form of combining quantum mechanics and general relativity, special relativity, and is the simplest unity between relativity and quantum, and corresponds to the extensive quantum theory [48-51]. It may overcome the singularity problem in general relativity, and is the combination and unification on quantum mechanics and general relativity [43].

4. Nonlinear Quantum Theory

The nonlinear approach of quantum mechanics is continuously an important direction. We propose mathematically the basic nonlinear operators [44,52]:

$$p_\mu=-i\hbar(F\frac{\partial}{\partial x_\mu}+i\Gamma_\mu), \qquad (16)$$

so Klein-Gordon equation and Dirac equations turn out to be

$$(F^2\partial_\mu^2+\Gamma_\mu^2-m^2)\varphi=-J. \qquad (17)$$

Dirac equations are:

$$\gamma_\mu(F\partial_\mu+i\Gamma_\mu)\psi+\mu\psi=j. \quad (18)$$

We obtained the corresponding Heisenberg equation. Then the present applied superposition principle is developed to the general nonlinear form. The quantum commutation and anticommutation belong to F and Γ_μ. This theory may include the renormalization, which is the correction of Feynman rules of curved closed loops. We think the interaction equations must be nonlinear. Many theories, models and phenomena are all nonlinear, for instance, soliton, nonabelian gauge field, and the bag model, etc [44,52]. The superluminal entangled state, which relates the nonlocal quantum teleportation and nonlinearity, should be a new fifth interaction. The relations among nonlinear theory and electroweak unified theory, and QCD, and CP nonconservation, etc., are expounded. Some known and possible tests are discussed [52].

Based on a hypothesis on duality exists probably only under interactions, so its test is that duality not exists when there are not interactions. Further, some nonlinear quantum equations and wave equations have soliton solutions, which correspond to particles. This is mathematical description of wave-particle duality, and whose condition is also interactions and nonlinearity. From the soliton-chaos double solutions of nonlinear equations, we proposed the field-particle duality, and field (wave)-quantum-chaos ternary. If we develop mathematical form, more multiplicity will be obtained. In the quantum entangled states, velocities of different phases obey possibly general Lorentz transformation (GLT) [53].

We proposed that a trefoil knot of left hand and a trefoil knot of right hand is possibly similar to two spins s=1/2 and -1/2 of three quarks (nucleons p and n, etc). Two topological structures should differ in their energy levels in the magnetic field. Basic mesons of two quark form a line, and s=0. This model is first applied to proton and neutron, then it may be extended to other particles [54].

5. Possible Unified Theory of Fermions and Bosons

It is known that fermion-pairs can form bosons. Further, any bosons can be composed of fermion-pairs. Therefore, fermions are more fundamental matter units, and bosons and fields can be composed of fermion-pairs. Such as p, e constitute hydrogen atoms, quark and antiquark pairs form mesons.

The time-independent Schrödinger equation is:

$$\Delta\psi+\frac{2m}{\hbar^2}(E-V)\psi=0. \tag{19}$$

Let $\frac{2m}{\hbar^2}(E-V)=k^2$, Eq.(19) becomes to the Helmholtz equation, i.e., wave equation:

$$\nabla^2\psi+k^2\psi=0. \qquad (20)$$

For the 3-dimensional central potential, let $\psi(r)=Y_{lm}(\Omega)u_{n,l}(r)/r$, Schrödinger equation is simplified to:

$$\{\frac{d^2}{dr^2}+\frac{2m}{\hbar^2}[E(n,l)-V(r)]-\frac{l(l+1)}{r^2}\}u_{n,l}(r)=0. \qquad (21)$$

The angular momentum quantum number l naturally produces a rotational potential.

The spin as part of the total angular momentum shows as the electron spin s is split to 2 in the magnetic field, which should develop Pauli equation, and in Eq.(20):

$$k^2=\frac{2m}{\hbar^2}E\pm\frac{2e}{c\hbar}(l+\frac{1}{2}). \qquad (22)$$

For different the angular momentum M and potential V substitution into the same k^2, the separation variables are still the same equation.

In spherical coordinates, by general method let $\psi=R(r)Y(\theta,\varphi)=R(r)\Theta(\theta)\Phi(\varphi)$,

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+[x^2-(j+\frac{1}{2})^2]y=0. \qquad (23)$$

It is j+(1/2) order Bessel equation, which is also discussed by Weinberg [55].

In column coordinates, by general method let $\psi=R(\rho)Z(z)\Phi(\varphi)$.

$$x^2\frac{d^2R}{dx^2}+x\frac{dR}{dx}+[x^2-m^2]R=0. \qquad (24)$$

R(x) is m order Bessel equation. If quantized j+(1/2) and m are all spins, Eqs.(23) and (24) will correspond to fermions and bosons, respectively.

We searched possible unified equations of fermions and bosons, both correspond to j+(1/2) order Bessel equation in spherical coordinates, and m order Bessel equation in column coordinates [56]. Further, this is corresponding relations between Bessel equation and spin:

spherical coordinate	j	fermions	Dirac equation	FD statistics
column coordinate	m	bosons	K-G equation	BE statistics, and BEC

In quark model both correspond to baryons with three quarks and mesons with quark-pair, which should be at column coordinates.

If both are related with topology, so fermions in the spherical coordinates are independent of each other, and corresponds to Pauli exclusion principle; bosons in the column coordinates are open, which may have multiple particles.

It is known that Kerr solution (1963) and Kerr-Newman solution (1965) have the column symmetry. Boyer-Lindquist (1967) and Carter (1968) completed Kerr solution and the maximum expansion of Kerr-Newman solution. This should correspond to my plane solution of general relativity [57]. If this is related to bosons, so it will be unstable. For evolution, it corresponds to the DNA. We should research the relations of DNA and bosons.

In general relativity, Schwarzschild solution and Reissner-Nordstrom solution have the spherical symmetry. It is relation to fermions. Except photon (s=1), only three fermions p, e, v are stable.

The Wheeler-De Witt functional differential equations with infinite degrees of freedom can not almost be solved [58]. The covariat quantization eventually abandoned the point particle model, and developed into string theory. The loop quantum gravity is found along the canonical quantization path.

Einstein thought that space and others all originate from the field, which corresponds to the bosons. While bosons can be produced by fermions, and formed in pairs (as photon γ=vv). Most basic graviton (spin s=2), photon γ(s=1), for strong and weak interactions (except π and K, s=0) gluon and W-Z are all s=1, which should be basis of field theory for the great unification of strong and weak electromagnetic forces.

Strong and weak interactions as short-range should be unified. Except different action ranges their main character is: strong interactions are attraction each other, and weak interactions are mutual repulsion and derive decay. We proposed a new method on their unification, whose coupling constants are negative and positive, respectively. Further, we researched a figure on the unification of the four basic interactions in three-dimensional space. So far the high energy experiments in the past sixty years have shown that the smallest mass fermions are proton, electron, neutrino and photon, which form the simplest model of particles. These fermions seem to be inseparable truth "atoms" (elements), because further experiments derive particles with bigger mass. They correspond to four interactions, and are also only stable particles. The final simplest theory is based on leptons (e- ν_e) and nucleons (p-n) or (u-d) in quark model with SU(2) symmetry and corresponding Yang-Mills field. Other particles and quark-lepton are their excited states. We discussed the simplest interactions and simplified QCD, and some possibly developed

directions of particle physics, for example, violation of basic principles, and precision and systematization of the simplest model, etc [59,60].

After three generations of quarks-leptons and Higgs boson have been confirmed, we researched some possible development directions of particle physics [61]. They include that particles of three generations are extended, the hadronic theory must precision quantification such as heavy flavor hadrons and the lifetime formula, etc., some basic principles (Pauli Exclusion Principle, duality, uncertainty principle, etc.) are probably violated under some cases. Santillli, Penrose, et al., proposed some new mathematical methods. Moreover, particle astrophysics, the negative matter as unified dark matter and dark energy [62-67] are searched.

Further, complex number, quaternion (whose unit constitutes SU(2) group), octonion [68], matrix and other number field or number system may be extended [69]. They may be applied to quantum theory and entanglement, communication [70-72], etc. We should research the entangled field is what field? Probably this is scalar, or vectors, tensor, spinor field, etc. It seems to be dual fields, inductive fields resonant to each other, and is the wave-field with the same frequency.

6. Mathematical Base of The Extensive Quantum Theories

According to Feynman's idea [73] and based on a new form of the Titius-Bode law [48,49]:

$$r_n = an^2, \qquad (25)$$

we developed a similar theory with the Bohr atom model, and obtain the quantum constants $H = (aGM_O)^{1/2}$ of the solar system and corresponding Schrödinger equation [48,49]:

$$iH\frac{\partial\psi}{\partial t} = -H^2\nabla^2\psi + (U - Q)\psi. \qquad (26)$$

Further, we proposed the extensive quantum theory in which the formulations are the same with the quantum mechanics and only quantum constant h and corresponding basic quantum elements are different [48-51].

The extensive quantum theory is based on the probability theory, and has statistical properties. It is related with fluid dynamics and its equations, and may be combined with Schrödinger equation, KG equation and Dirac equations. This is combined with the probability equation, and corresponds to synergetics. The system of natural units (c=h=1) is namely a unified extensive form.

The extensive quantum theory has wave, interference, superposition, entanglement, coherence and interaction, etc. Coherence, entanglement, and interactions of macromolecules may produce biology, life, A-T, G-C, and DNA double helix structures, etc.

Further, we researched, the extensive special relativity [44,74] and the extensive general relativity [75], etc. These theories representations are the same with those original theories, only the basic constants can be different. Mathematical bases of various extensive theories are all the isomorphism, and may be based on the system theory, symmetry, fractal, etc. The scaling invariance is mathematical basis of various extensive theories. This should have the corresponding renormalization group equation, in the original equation, , m, etc., transform to c, h, etc.

7. Some Possible Developments of Quantum Theory

The essence of the renormalization is to eliminate the infinite background, i.e., related to the vacuum and the infinite potential. These and the renormalization are not required in the gravitational field.

It is known that baryon is Dirac equation, and meson is Klein-Gordon equation. Both are approximately Schrödinger equation. This may obtain $r_n = an^2$, and corresponding equations may derive $p_n = bn^2$, and $E_n = cn^2 = \frac{p^2}{2m}$, $p = Dn$, etc.

Weinberg proposed that we should take it seriously and find a more satisfying other possible theory, and quantum mechanics is only a good approximation of it [55].

We discussed possible 16 types developments of quantum equations [56].

Life changes with energy. There are vibrations, such as resonant or nonresonant, and rotation. It can be used to study life, its spectrum and quantized.

Space-time has an uncertainty relation, and is quantization. Combined with the logical structure of quantum mechanics [76], it may have statistics. p and r conjugate, E and t conjugate, M_z and φ conjugate. Three are symmetry, and form uncertainty relations each other.

This method can be generalized to the mechanical wave theory [77], and obtain the conjugate equation, and the mass-lifetime formula, etc.

The three-generation quark equations may be unified. Only n, l and Y, Q are different, except u, d, the other I all is 0. There have relations with P, C, PC, and PCT.

The relativity equation is:

$$s^2 = T^2 - R^2, T^2 = s^2 + R^2. \qquad (27)$$

$$T^2\psi = -\hbar^2 \frac{\partial^2}{\partial E^2}\psi = (s^2 + R^2)\psi. \qquad (28)$$

$$\frac{\partial^2}{\partial p_\mu^2}\psi = [\frac{\partial^2}{\partial p^2} - \frac{\partial^2}{\partial E^2}]\psi = \frac{s^2}{\hbar^2}\psi. \qquad (29)$$

It is similar Klein-Gordon equation. The similar Dirac equations are:

$$i\hbar\frac{\partial}{\partial E}\psi = (\alpha s + \beta R)\psi. \qquad (30)$$

$$\gamma_\mu \frac{\partial}{\partial p_\mu}\psi = s\psi. \qquad (31)$$

It is replaced by various space-time metric, and can obtain various equations of similar quantum mechanics, such as for the vacuum $s^2 = 0$.

We should research equation and so on for anyon.

It is known that the charged Dirac equations become non-relativistic, so can obtain spin. By similar way some effect should be obtained. But this must first consider the change in space-time in the electromagnetic field, as for Reissner-Nordstrom metric

$$g_{00}c^2dt^2 = (1 - \frac{2m}{r} + \frac{4\pi GQ^2}{c^4r^2})c^2dt^2. \qquad (32)$$

The equation with charge is:

$$(\gamma_\mu \frac{\partial}{\partial p_\mu} - s + AQ^n)\psi = 0. \qquad (33)$$

This should combine the electromagnetic general relativity [78].

When time has a direction, the corresponding equations may be developed. At this time the time may be the vector, and corresponds to the 3 dimension time [79]:

$$-\hbar^2\Delta\psi = \vec{t}\,\psi. \qquad (34)$$

It can introduce the spherical coordinates, and obtain $\frac{l(l+1)}{r^2}$.

This follows that space-time changes with energy, and consistent with

general relativity. For general relativity, $s^2 = g_{\mu\nu}x_\mu x_\nu$,

$$x_\mu \psi = i\hbar \frac{\partial}{\partial p_\mu}\psi = (s^2 / g_{\mu\nu}x_\nu)\psi. \qquad (35)$$

General case is:

$$s^2\psi = g_{\mu\nu}x_\mu x_\nu \psi = -\hbar^2 g_{\mu\nu}\frac{\partial^2}{\partial p_\mu \partial p_\nu}\psi. \qquad (36)$$

There is the electromagnetic field, it is:

$$g_{\mu\nu}x_\mu x_\nu \psi + \hbar^2 g_{\mu\nu}\frac{\partial^2}{\partial p_\mu \partial p_\nu}\psi + A(Q^2 + V)\psi = 0. (37)$$

For special relativity,

$$s^2\psi = [(ct)^2 - r^2]\psi = -\hbar^2[\frac{\partial^2}{\partial(E/c)^2} - \frac{\partial^2}{\partial p^2}]\psi. \quad (38)$$

Various metrics, as Schwarzschild metric

$$(1 - \frac{2m}{r})c^2 dt^2. \qquad (39)$$

$$T^2\psi = (1 - \frac{2m}{r})(-\hbar^2 \frac{\partial^2}{\partial E^2})\psi. \qquad (40)$$

$$(1 - \frac{2m}{x_\mu})x_\mu^2\psi = (x_\mu^2 - 2mx_\mu)\psi = (-\hbar^2\frac{\partial^2}{\partial p_\mu^2} - 2im\hbar\frac{\partial}{\partial p_\mu})\psi. (41)$$

This equations have the first-order and second-order differential. Reissner-Nordstrom metric is $\frac{R^2(t)}{1-kr^2}dr^2$. $g_{\mu\nu}$ correspond to E and V.

We can further unify quantum theory and general relativity. Relativity focuses on space-time, while quantum theory focuses on matter and mass-energy. This can particularly describe the quantized discrete space-time, in which dx_μ is a quantum space-time. We proposed the general equations are [43]:

$$G_{\mu\nu}\psi = \kappa T_{\mu\nu}\psi. \qquad (42)$$

This should be replaced by the operator representations of $G_{\mu\nu}$ or/and $T_{\mu\nu}$ in the energy-momentum and space-time. It can be related to the Wheeler-de Witt cosmic equations of quantum mechanics.

It can be similarly developed by method of nonlinear quantum theory

[44,52], in which F corresponds to $g_{\mu\nu}$, and Γ corresponds to potential V.

In astronomy and cosmology, it corresponds to general relativity. Such various models of cosmic variation can be obtained, as the Big Bang, inflation, the circular universe, etc.

Charged particles may combine the cosmical electrodynamics [79].

Combining extensive quantum theory [48-51], the possible wave functions must also be generalized to extensive quantum theory.

In statistical mechanics and entropy,

$$S\psi = \frac{E}{T}\psi = i\hbar\frac{1}{T}\frac{\partial}{\partial t}\psi. \tag{43}$$

The simplest is h→h/T.

Based on the Noether theorem, energy E is related to time t: 1). E is conserved, t is uniform; E is constant, and t is also constant. 2). E change, t has direction; especially when dE/T=dS. 3). More generally, when various changes, t shows the direction. 4). E and various cycles, t also cycle, or spiral rise. 5). E slows, as does t and longevity. 6). When chaotic and nonlinear change, t is not uniform and has very critical moments.

Wendt studied systematically quantum mind and social science, and proposed that people are the walking wave functions based on the causal closure (or completeness) of physics (CCP) [80]. This research core is that quantum theory can explain the phenomena of consciousness and intentionality, and an important basis is the quantum entanglement and macroscopic entanglement. We discussed quantum sociology, whose bases are the extensive quantum theory and the social individual-wave duality [81,82]. Quantum may be extended to people, animals and plants, all the life span, evolution, etc. V (E) is a factor affecting life, time and its structure, and is external factor and external potential. Infinite deep potential trap, i.e., any life is all limited, and different people have different energy levels. It may combine Maslow's different needs.

In a word, quantum theory and its mathematics may be applied to many aspects. It will be useful and very meaning.

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